Self-adaptive proximal algorithms for equilibrium problems in Hadamard space

https://doi.org/10.31713/MCIT.2025.081

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Abstract—**We consider a new self-adaptive algorithms for equilibrium problem in Hadamard spaces. At each step of the algorithms, the sequential minimization of two special strongly convex functions is performed. Our self-adaptive algorithms do not calculate bifunction values at additional points and do not require knowledge of bifunctions' Lipschitz constants. For pseudomonotone bifunctions of Lipschitz-type, theorems on weak convergence of sequences generated by algorithms are proved.**

Keywords—Hadamard space, equilibrium problem, algorithms, convergence

I. Introduction

There are many problems arising from applied nonlinear analysis that can be modeled as an equilibrium problem. The equilibrium problem includes as particular cases, optimization problems, variational inequalities, Nash equilibrium problems, as well as some other problems of interest in many applications [1, 2]. The formulation of the equilibrium problem was first given in the works of H. Nikaido and K. Isoda in the 1950s [3]. These works are devoted to proving the existence of Nash equilibrium points in non-cooperative games. The current stage in the study and application of equilibrium problems began with the article [4]. The study of algorithms for solving equilibrium problems is actively continuing. Recently, motivated by problems of Machine Learning, there has been a need to develop theory and algorithms for solving mathematical programming problems in metric Hadamard spaces (also known as CAT(0) spaces) [5]. Another strong motivation for studying these problems is the possibility of writing certain nonconvex problems in geodesically convex form in a space with a specially chosen Riemannian metric [6]. A notable interest has arisen in equilibrium problems in Hadamard spaces [7, 8].

In this paper, we consider a new self-adaptive algorithms for equilibrium problem in Hadamard spaces. At each step of the algorithms, the sequential minimization of two special strongly convex functions is performed. Our self-adaptive algorithms do not calculate bifunction values at additional points and do not require knowledge of bifunctions' Lipschitz constants.

Details about Hadamard spaces can be found in [5].

II. Equilibrium problem in a Hadamard space

Let (X,d) be a Hadamard space. Consider the equilibrium problem for nonempty closed convex set $C \subseteq X$ and bifunction $F : C \times C \to R$:

$$\text{find } x \in C \text{ such that } F(x,y) \ge 0 \quad \forall y \in C . \quad (1)$$

Suppose that the following conditions hold:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) functions $F(x,\cdot) : C \to R$ are convex and lower semicontinuous for all $x \in C$;

(A3) functions $F(\cdot,y) : C \to R$ are upper weakly semicontinuous for all $y \in C$;

(A4) bifunction $F : C \times C \to R$ is pseudomonotone, i.e. for all x, $y \in C$ from $F(x,y) \ge 0$ it follows that $F(y,x) \le 0$.

(A5) bifunction $F : C \times C \to R$ is Lipschitz type, i.e. there exist $a > 0$, $b > 0$, such that $\forall x, y, z \in C$

$$F(x,y) \le F(x,z) + F(z,y) + ad^2(x,z) + bd^2(z,y) .$$

If $F(x,y) = (Ax, y - x)$, where $A : C \to H$, C is nonempty subset of Hilbert space H, then problem (1) takes form of variational inequality

$$\text{find } x \in C \text{ such that } (Ax, y - x) \ge 0 \quad \forall y \in C .$$

Henceforth, we assume that $S \neq \varnothing$.

III. SELF-ADAPTIVE PROXIMAL ALGORITHMS

For approximate solution of (1) we consider extraproximal algorithm with adaptive choice of step size [9].

Algorithm 1. Choose element $x_1 \in C$, $\tau \in (0,1)$, $\lambda_1 \in (0,+\infty)$. Set $n = 1$.

Step 1. Compute

$$y_n = \arg\min_{y \in C} \left(F(x_n, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right).$$

If $x_n = y_n$, then stop and $x_n \in S$. Otherwise, go to step 2.

Step 2. Compute

$$x_{n+1} = \arg\min_{y \in C} \left(F(y_n, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right).$$

Step 3. Compute

$$\lambda_{n+1} = \begin{cases} \lambda_n, \text{ if } \quad F(x_n, x_{n+1}) - F(x_n, y_n) - F(y_n, x_{n+1}) \le 0, \\ \min\left\{ \lambda_n, \dfrac{\tau}{2} \dfrac{d^2(x_n, y_n) + d^2(x_{n+1}, y_n)}{\left(F(x_n, x_{n+1}) - F(x_n, y_n) - F(y_n, x_{n+1}) \right)} \right\}, \text{ otherwise.} \end{cases}$$

Set $n := n+1$ and go to step 1.

On each step of Algorithm 1 we need to solve two convex problems with strongly convex functions. No information about constants a and b is used.

Obviously the sequence (λ_n) is nonincreasing. Also it is lower bounded by $\min\{\lambda_1, \tau/2\max\{a,b\}\}$.

Let us prove an important estimate relating the distances between the points generated by Algorithm 1 to an arbitrary element of the set of solutions S.

Lemma 1. For the sequences (x_n) and (y_n), generated by Algorithm 1, the following inequality holds:

$$d^2(x_{n+1}, z) \le d^2(x_n, z) - \left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) - \left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(y_n, x_n),$$

where $z \in S$.

Let us formulate one of the main results.

Theorem 1 ([9]). Let (X,d) be an Hadamard space, $C \subseteq X$ be a nonempty convex closed set, for bifunction $F : C \times C \to R$ conditions A1–A5 are satisfied and $S \neq \varnothing$. Then sequences (x_n), (y_n) generated by Algorithm 1 converge weakly to the solution $z \in S$ of equilibrium problem (1), moreover, $\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d(y_n, x_{n+1}) = 0$.

In paper [10] for solution of problem (1) the following algorithm was proposed

$$\begin{cases} y_n = \arg\min_{y \in C} \left(F(y_{n-1}, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right), \\ x_{n+1} = \arg\min_{y \in C} \left(F(y_n, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right), \end{cases} \qquad (2)$$

where values $\lambda_n > 0$ were set according to the requirement

$$\{\inf_n \lambda_n, \sup_n \lambda_n\} \subseteq \left(0, \frac{1}{2(2a+b)}\right),$$

i.e. the information about constants from condition (A5) was used. Based on the scheme (2) and paper [11], we construct a two-stage proximal algorithm with adaptive choice of the value λ_n.

Algorithm 2. Choose elements x_1, $y_0 \in C$, $\tau \in (0, \frac{1}{3})$, $\lambda_1 \in (0,+\infty)$. Set $n = 1$.

Step 1. Compute

$$y_n = \arg\min_{y \in C} \left(F(y_{n-1}, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right).$$

Step 2. Compute

$$x_{n+1} = \arg\min_{y \in C} \left(F(y_n, y) + \tfrac{1}{2\lambda_n} d^2(y, x_n) \right).$$

If $x_{n+1} = x_n = y_n$, then stop and $x_n \in S$. Otherwise, go to step 3.

Step 3. Compute

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \quad F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1}) \le 0, \\ \min\left\{ \lambda_n, \dfrac{\tau}{2} \dfrac{d^2(y_{n-1}, y_n) + d^2(x_{n+1}, y_n)}{\left(F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1}) \right)} \right\}, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to the step 1.

Let us present the main results on the convergence of the Algorithm 2.

Lemma 2 ([12]). For the sequences (x_n) and (y_n), generated by Algorithm 2, the following inequality holds:

$$d^2(x_{n+1}, z) \le d^2(x_n, z) - \left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) - \left(1 - 2\tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(y_n, x_n) + 2\tau \frac{\lambda_n}{\lambda_{n+1}} d^2(x_n, y_{n-1}),$$

where $z \in S$.

Theorem 2 ([12]). Let (X,d) be a Hadamard space, $C \subseteq X$ be a nonempty convex closed set, for bifunction $F : C \times C \to R$ conditions A1–A5 are satisfied and $S \neq \varnothing$. Then sequences (x_n), (y_n), generated by Algorithm 2, converge weakly to the solution $z \in S$ of problem (1).

In the future, we plan to consider more specialized versions of the Algorithm 2 for variational inequalities and min-max problems on Hadamard manifolds (for example, on the manifold of symmetric positive definite matrices). Constructing randomized adaptive versions of algorithms is also an interesting research direction.

IV. Conclusions

We considered equilibrium problems in metric Hadamard spaces. For an approximate solution of equilibrium problems, iterative two-stage proximal algorithms have been proposed and studied. At each step of the algorithms, the sequential minimization of two special strongly convex functions should be done. Our self-adaptive algorithms do not calculate bifunction values at additional points and do not require knowledge of bifunction's Lipschitz constants. For pseudo-monotone bifunctions of the Lipschitz type, weakly upper semicontinuous in the first variable, convex and lower semicontinuous in the second variable, theorems establishing the convergence of sequences generated by the algorithms are proved.

A direction for future research could be the exploration of possible extensions of our results to Banach spaces [13] and stochastic setup [14]. And of course the question of the rate of convergence under stronger assumptions is very important.

Acknowledgment

This work was supported by the NAS of Ukraine (project “New subgradient and extragradient methods for non-smooth regression problems”, 0124U002162).

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