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Numerical Optimization of a Pseudoparabolic Systems with Memory

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Abstract—**We employ the method of *a priori* inequalities in negative norms to prove the existence of a pointwise optimal control for the regularized problem corresponding to a time-nonlocal pseudoparabolic integro-differential equation with a Volterra-type integral term. Certain differential properties of the cost functional are investigated, and a numerical example illustrating the computation of the optimal control is presented.**

Keywords— Dirichlet problem; integro-differential equation; pseudoparabolic equation, Volterra operator, a priori estimates, generalized solutions, optimal control

I. Introduction

Within applied mathematics, considerable interest is directed toward pseudoparabolic differential and integro-differential formulations. Equations of this character serve as analytical frameworks for a broad spectrum of physical phenomena, encompassing delayed radiation transport, two-phase flow in porous structures influenced by dynamic capillarity or hysteretic behavior, ionic diffusion within soils, and thermal propagation through heterogeneous composites (see, for instance, [1], [2], and sources cited therein).

In [1], [3] S.I. Lyashko and his collaborators developed the method of *a priori* inequalities in negative norms, which enabled the study of a broad class of optimization problems for systems with distributed parameters, including differential models of the pseudoparabolic type (see also [4] and the references therein). It was later established that this methodology is also well suited for addressing Dirichlet problems involving integro-differential equations with Volterra-type integral terms [2], [5], [6].

We consider the linear pseudoparabolic differential equation $\mathcal{L}u \equiv \mathcal{L}_1 u + \mathcal{L}_2 u = f$, where

$$\mathcal{L}_1 u \equiv -\sum_{i,j=1}^{n}\left(a_{ij}(x)\,u_{x_j}\right)_{x_i t} + a(x)u_t - \\ -\sum_{i,j=1}^{n}\left(b_{ij}(x)\,u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} b_i\,(x)u_{x_i} + b(x)u, \quad (1)$$

and

$$\mathcal{L}_2 u \equiv \int_0^t \sum_{i=1}^{n}\left(K_{ij}(x,t,\tau)\,u_{x_i}(x,\tau)\right)_{x_i} + \\ \sum_{i=1}^{n} K_i\,(x,t,\tau)u_{x_i} + K(x,t,\tau)u\,d\tau, \quad (2)$$

with initial-boundary Dirichlet conditions

$$u|_{t=0} = 0,\ u|_{x\in\partial\Omega} = 0. \quad (3)$$

In [6], we stated a priori inequalities similar to those in [1] and [2] for the case $b_i = K_i = K = 0$. The results concerning the well-posedness of the initial-boundary problems and the existence of optimal control was justified in the cited work.

II. Main Spaces

The evolution of the system is described by the linear equation $\mathcal{L}u = f$ in the domain $Q = \Omega \times (0,T)$. We assume some smoothness assumptions on the coefficients and the kernels as in [6].

By W_{BR}, H_{BR} we denote the completion of the space of smooth functions C_{BR}^{∞} that satisfy the conditions (3) with respect to the norms

$$\| u \|_{W_{BR}} = \left(\int_Q u_t^2 + \sum_{i=1}^{n} u_{x_i t}^2\,dQ\right)^{\frac{1}{2}}, \quad (4)$$

$$\| u \|_{H_{BR}} = \left(\int_Q u^2 + \sum_{i=1}^{n} u_{x_i}^2\,dQ\right)^{\frac{1}{2}}. \quad (5)$$

Similarly, we define W_{BR^+}, H_{BR^+}. Finally, by W_{BR}^-, H_{BR}^-, $W_{BR^+}^-$, $H_{BR^+}^-$ we denote the negative spaces constructed from the corresponding positive spaces with respect to $L_2(Q)$.

Paper [6] provides theorems on well-posedness and existence on optimal control. In particular

Definition. *A function $u \in H_{BR}$ is called a generalized (weak) solution of the equation $\mathcal{L}u = f$, $f \in W_{BR^+}^-$ if*

$$\langle u, \mathcal{L}^* v\rangle_{H_{BR}} = \langle f, v\rangle_{W_{BR^+}},$$

for any functions $v \in W_{BR^+}$ such that $\mathcal{L}^* v \in H_{BR}^-$.

Theorem 1. *For any $f \in W_{BR^+}^-$ there exists a unique solution $u \in H_{BR}$ of the equation $\mathcal{L}u = f$ in the sense of definition above, and there exists a constant $C > 0$ such that $\| u \|_{H_{BR}} \leq C \| f \|_{W_{BR^+}^-}$.*

III. Control Regularization

We average the right-hand side $F = f + Ah$ in optimal control problem from [6] and consider the regularized pointwise control problem:

$$\mathcal{L}u_\varepsilon = f_\varepsilon + \mathcal{A}_\varepsilon h, u_\varepsilon \in W_{\mathrm{BR}}, \quad (6)$$

$$J_\varepsilon(h) = ||u_\varepsilon - u(h)||^2 . \quad (7)$$

Here,

$$f_\varepsilon(x,t) = \int \omega_\varepsilon(y) f(x_1 - y, x_2, .., x_{\mathrm{n}}, t)dy, \quad (8)$$

$$\mathcal{A}_\varepsilon h(x,t) = \sum_{k=1}^{s} a_\varepsilon(x_1 - x_{1,\mathrm{k}})\varphi_{\mathrm{k}}(x_2, .., x_{\mathrm{n}}, t). \quad (9)$$

with appropriate convolutional kernels ω_ε and $a_\varepsilon = \frac{1}{\varepsilon} 1_{[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]}$.

It can be shown that, if the set of admissible controls $\mathcal{U}$ is closed, convex, and bounded in the Hilbert space $\mathcal{H}$, then the optimal control problem (6)-(9) has solution, which converges weakly to the solution of the initial optimal control problem in the space $\mathcal{H}$, when $\varepsilon \to 0$

Theorem 2. *For the problem (6)-(9), the functional $J_\varepsilon(h)$ is Fréchet differentiable for $h \in \mathcal{U}$, and its Fréchet derivative has the form:*

$$J_h(\Delta h) = \frac{1}{\varepsilon} \int_{Q'} \sum_{i=1}^{s} (Rv)\, \varphi_i(x', t) dQ' \cdot \Delta x_{1,i} +$$

$$+ \frac{1}{\varepsilon} \int_{Q'} \sum_{i=1}^{s} \int_{x_{1,i}-\varepsilon/2}^{x_{1,i}+\varepsilon/2} v(\xi, x', t) d\xi \cdot \Delta\varphi_i(x', t) dQ',$$

where

$$Rv = v\left(x_{1,i} + \frac{\varepsilon}{2}, x', t\right) - v\left(x_{1,i} - \frac{\varepsilon}{2}, x', t\right),$$

and v is the solution of the adjoint problem

$$\mathcal{L}^* v(h) = 2(u(h) - z_0).$$

IV. Numerical example

Let us consider $\Omega = [-1,1]$, $K_{11} = 1, K_1 = K = 0$, $a_1(x) = 1$, $b(x) = -1$, $b_1(x) = 0$, $b_{11}(x) = -1$, $T = 1$.

Thus, the regularized problem under consideration is of the form:

$$u_t - u_{txx} - u + u_{xx} + \int_0^t u_{xx}(x, \tau) d\tau = a_\varepsilon(x - x_1)\varphi(t).$$

Here, $x \in \Omega = [-1,1], t \in [0,1]$, and (x_1, φ) is the control that belongs to a bounded set of admissible controls $\mathcal{U}$ from the control space $\mathcal{H} = [-1,1] \times L_2(0,1)$. $\mathcal{U}$ is assumed to be sufficiently large. To find the optimal control, we will use the gradient method: $h_{i+1} = h_i - \mu_i \cdot \mathrm{grad} J_{h_i}$. For the initial approximation, we use $x_1 = 0, \varphi(t) = 0$. At each step of the gradient method, the gradient of the quality functional $\mathrm{grad} J_{h_i}$ is computed using the previously computed Fréchet derivative. To compute this gradient, the forward and adjoint problems must be solved

$$\mathcal{L}u(h_i) = \mathcal{A}_\varepsilon h_i, \mathcal{L}^* v(h) = 2(u(h) - z_0).$$

The solutions to the latest problems are obtained using implicit finite-difference schemes.

TABLE I. Table of Values of The Control and The Cost Functional for Different Numbers of Iterations.

Iteration	x_1	J	Iteration	x_1	J
0	0	1.2157	5200	0.4728	0.0231
200	0.2613	0.5954	5400	0.4633	0.0217
400	0.3020	0.3962	5600	0.4692	0.0205
600	0.3324	0.2928	5800	0.4756	0.0193
800	0.3595	0.2296	6000	0.4687	0.0185
1000	0.3709	0.1869	6200	0.4751	0.0173
1200	0.3865	0.1562	6400	0.4711	0.0165
1400	0.3951	0.1331	6600	0.4794	0.0157
1500	0.4030	0.1234	6800	0.4727	0.0150
1600	0.4032	0.1149	7000	0.4716	0.0143
1800	0.4156	0.1004	7200	0.4815	0.0138
2000	0.4213	0.0886	7400	0.4746	0.0132
2200	0.4282	0.0788	7600	0.4744	0.0127
2400	0.4303	0.0705	7800	0.4847	0.0121
2600	0.4356	0.0635	8000	0.4792	0.0117
2800	0.4397	0.0576	8200	0.4787	0.0113
3000	0.4406	0.0522	8400	0.4841	0.0108
3200	0.4406	0.0477	8600	0.4895	0.0104
3400	0.4525	0.0438	8800	0.4833	0.0100
3600	0.4541	0.0402	9000	0.4845	0.0097
3800	0.4547	0.0371	9200	0.4883	0.0093
4000	0.4599	0.0344	9400	0.4892	0.0090
4200	0.4549	0.0320	9600	0.4855	0.0088
4400	0.4571	0.0299	9800	0.4906	0.0088
4600	0.4687	0.0278	10000	0.4836	0.0083
4800	0.4608	0.0260	10100	0.4863	0.0081
5000	0.4669	0.0244			

In the graphs below, we will display the results of the numerical approximation $u(h)$ and $\varphi(t)$ after $n = 400, 1500, 4000, 10000$ iterations and plots the change of the cost functional $J(h)$ as well as the graph of the change in the control with respect to the control variable x_1.

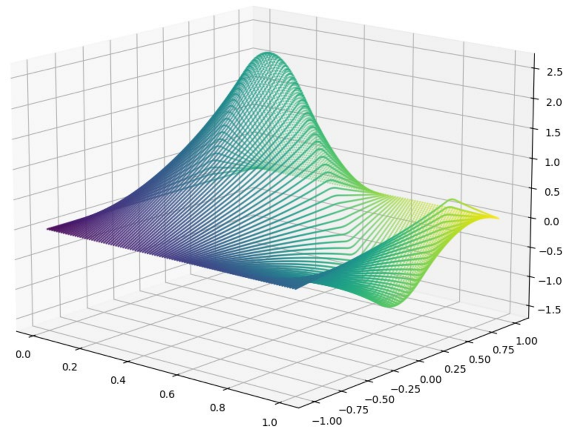

Figure 1: Desired state function z_0 .

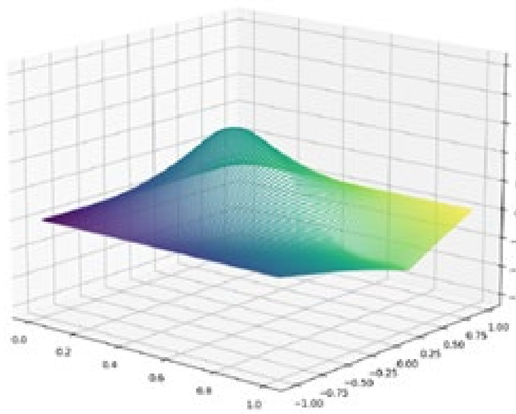

Figure 2: Numerical approximations $u(h)$ after 400 iterations.

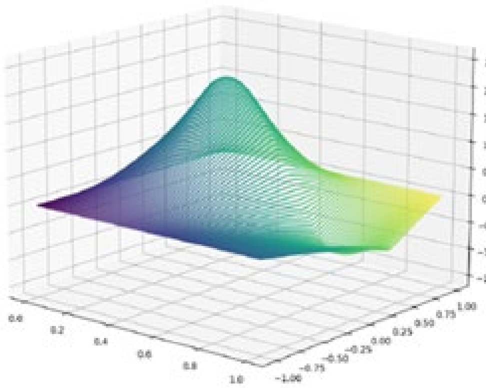

Figure 3: Numerical approximations $u(h)$ after 1500 iterations.

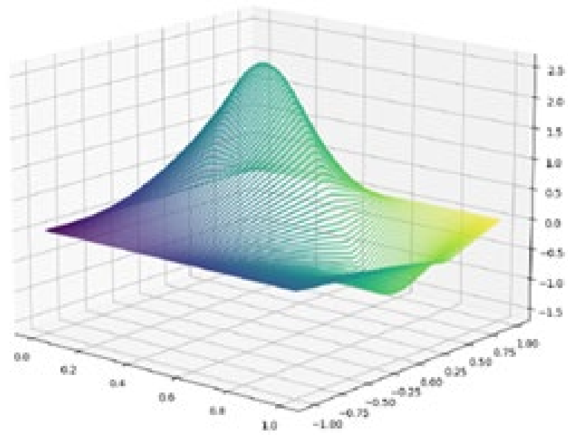

Figure 4: Numerical approximations $u(h)$ after 4000 iterations.

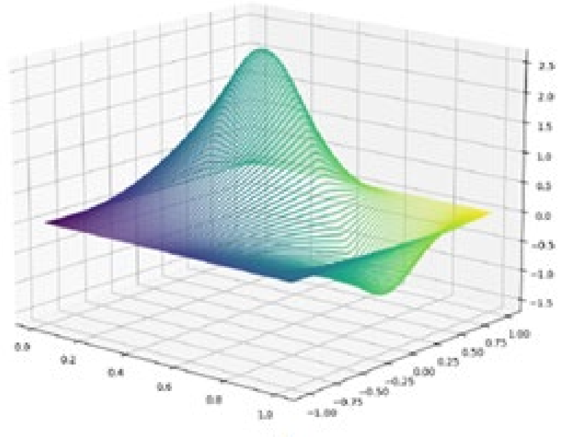

Figure 5: Numerical approximations $u(h)$ after 10000 iterations.

V. CONCLUSIONS

In this paper, we examine the problem of determining an optimal pointwise control for systems governed by pseudoparabolic equations, integro-differential in nature. The study is grounded in the method of *a priori* estimates in negative norms. By establishing such inequalities within an appropriate system of functional spaces, we claim the existence of an optimal control for both the original and the regularized formulations. Unlike many related studies, our approach does not rely on the nonnegative definiteness of the associated differential operator, thereby expanding the scope of its applicability.

The derived Fréchet derivative of the cost functional for the regularized problem enables the use of gradient-based numerical schemes to compute the optimal control. A representative example demonstrates the effectiveness of this method: by employing a regularized point control, we achieve close agreement between the computed and desired trajectories, as confirmed by the diminishing value of the cost functional. Additional numerical results and visualizations highlight the convergence behavior,

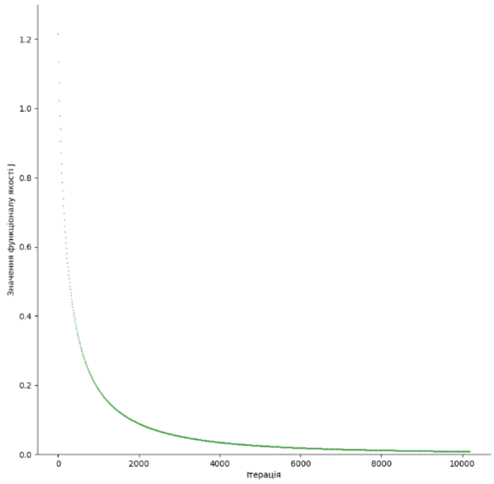

Figure 6: The graph of the change in the costcriterion.

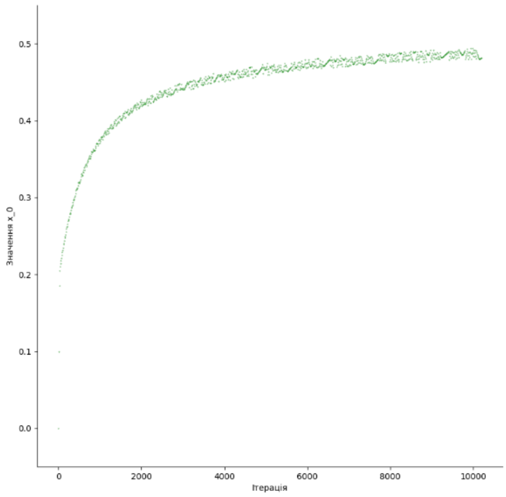

Figure 7: The graph of the change in the control.

including a non-monotonic trend in one component of the control, likely due to the gradient step size. Although reducing the step could improve smoothness, it would significantly increase computational effort.

It is important to note that the cost functional is generally non-convex, and thus the resulting control may represent either a local or a global minimum. This issue deserves further exploration in specific contexts. Nevertheless, in the presented example, convergence of the cost functional to zero confirms the attainment of an optimal control.

A natural continuation of this work involves the study of controllability and asymptotic controllability problems for the corresponding systems, which may further extend the theoretical and practical implications of the proposed framework.

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