On the Solution of the Linear Set-valued Homogeneous Cauchy Problem with the Hukuhara Derivative

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Abstract—**The paper considers a linear set-valued homogeneous Cauchy problem with the Hukuhara derivative and provides its analytical solution.**

Keywords—linear; Cauchy problem; se-valued mapping; Hukuhara derivative

I. Introduction

As is well established, the theory of set-valued differential equations finds extensive application in control theory, the theory of differential inclusions, and fuzzy systems (see [1–6] and references therein). At first sight, such equations resemble their classical counterparts; however, their analysis and solution inevitably require explicit consideration of their set-valued character. Consequently, classical methods and approaches developed for single-valued systems cannot, in general, be applied directly to the set-valued case, thereby necessitating the development of new or adapted methodologies. Moreover, the inherent set-valued structure of these equations gives rise to distinctive properties that warrant systematic investigation.

The report presents an analytical form of the solution for a linear homogeneous differential equation with the Hukuhara derivative.

II. Preliminaries

Let R be the set of real numbers and R^n be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(R^n)$ the set of nonempty compact and convex subsets of R^n.

For two given sets $X, Y \in conv(R^n)$ and $\lambda \in R$, the Minkowski sum and scalar multiple are defined by $X + Y = \{x + y \mid x \in X, y \in Y\}$ and $\lambda X = \{\lambda x \mid x \in X\}$.

The following properties hold:

1) $X + Y = Y + X \in conv\left(R^n\right)$, $\lambda X \in conv\left(R^n\right)$;
2) If $\alpha\beta \geq 0$, then $\alpha X + \beta X = \left(\alpha + \beta\right) X$;
3) $\lambda X + \lambda Y = \lambda\left(X + Y\right)$.

Also, let's add one more operation: the product of a matrix with a set $AX = \bigcup_{x \in X} Ax$, where $A \in R^{n \times n}$ is real matrix of size $n \times n$ and $X \in conv\left(R^n\right)$.

We will list some properties of this operation:

1) If $A \in R^{n \times n}$ and $X \in conv(R^n)$, then $AX \in conv\left(R^k\right)$, where $k = rank(A)$;
2) If $A \in R^{n \times n}$ and $X, Y \in conv(R^n)$, then $AX + AY = A\left(X + Y\right)$;
3) If $A, B \in R^{n \times n}$ and $X \in conv(R^n)$, then $\left(A + B\right) X \subseteq AX + BX$;
4) If $A \in R^{n \times n}$, $X, Y \in conv(R^n)$ and $X \subseteq Y$, then $AX \subseteq AY$.

Consider the Pompeiu-Hausdorff distance $h(\cdot,\cdot)$ given by

$$h(X,Y) = \min\left\{r \geq 0 \mid X \subset Y + B_r(0),\ Y \subset X + B_r(0)\right\},$$

where $B_r(0) = \left\{x \in R^n \mid \|x\| \leq r\right\}$ is the closed ball with radius r centered at the origin ($\|x\|$ denotes the Euclidean norm).

It is known that $(conv(R^n), h)$ is a complete metric space. However, $conv(R^n)$ is not a linear space since it does not contain inverse elements for the addition, and therefore difference is not well defined, i.e. if $A \in conv(R^n)$ and $A \neq \{a\}$, then $A + (-1)A \neq \{0\}$. As a consequence, alternative formulations for difference have been suggested. One of these alternatives is the Hukuhara difference [7].

Let $X, Y \in conv(R^n)$. A set $Z \in conv(R^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X\overset{H}{-}Y$.

In this case $X\overset{H}{-}X=\{0\}$ and also $(A+B)\overset{H}{-}B=A$ for any $A,B\in conv(R^n)$.

Simultaneously, M. Hukuhara introduced the concept of H-differentiability [7] for set-valued mappings by using the H-difference.

Definition 1 [7]. Let $X:[0,T]\to conv(R^n)$ and $t\in[0,T]$. We say that $X(\cdot)$ has a H-derivative $D_H X(t)\in conv(R^n)$ at $t\in(0,T)$, if for all $\Delta>0$ that are sufficiently close to 0, the H-differences and the limits exist

$$\lim_{\Delta\to 0}\Delta^{-1}(X(t+\Delta)\overset{H}{-}X(t))=$$

$$\lim_{\Delta\to 0}\Delta^{-1}(X(t)\overset{H}{-}X(t-\Delta))=D_H X(t).$$

Theorem 1 [7]. If the mapping $X:[0,T]\to conv(R^n)$ is H-differentiable on $[0,T]$, then $X(t)=X(0)+\int_0^t D_H X(s)ds$, where the integral is understood in the sense of [7].

Corollary 1. If the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$, then $diam(X(\cdot))$ is a non-decreasing function on $[0,T]$.

Corollary 2. If the function $diam(X(\cdot))$ is a decreasing function on $[0,T]$, then the set-valued mapping $X(\cdot)$ is not H-differentiable on $[0,T]$.

III. Linear Cauchy Problem

Now, consider the Cauchy problem

$$D_H X(t)=A(t)X(t),\quad X(0)=X_0, \tag{1}$$

where $X:[0,T]\to conv\left(R^n\right)$ is the unknown set-valued mapping, $A:[0,T]\to R^{n\times n}$ is a continuous matrix function and $\det\left(A(t)\right)\neq 0$ for all $t\in[0,T]$.

The set-valued mapping $X(\cdot)$ will be called the solution of the system (1) on the interval $[0,T]$ if it is continuously and satisfies system (1) on $[0,T]$.

Theorem 2. System (1) has a unique solution of the form

$$X(t)=X_0+\sum_{k=1}^{\infty}\left[\int_0^t A(\tau_k)\int_0^{\tau_k}A(\tau_{k-1})\ldots\int_0^{\tau_2}A(\tau_1)X_0 d\tau_1\ldots d\tau_k\right]$$

for all $t\in[0,T]$.

Remark. For all $k\ge 1$ and $t\in[0,T]$

$$\int_0^t A(\tau_k)\int_0^{\tau_k}A(\tau_{k-1})\ldots\int_0^{\tau_2}A(\tau_1)d\tau_1\ldots d\tau_k X_0\subseteq$$

$$\int_0^t A(\tau_k)\int_0^{\tau_k}A(\tau_{k-1})\ldots\int_0^{\tau_2}A(\tau_1)X_0 d\tau_1\ldots d\tau_k.$$

Remark. If the matrices $A(t)$ and $A(s)$ are commutative matrices for all $t,s\in[0,T]$, i.e. the equality $A(t)A(s)=A(s)A(t)$ holds for all $t,s\in[0,T]$, then

$$X(t)=X_0+\sum_{k=1}^{\infty}\int_0^t\frac{\left(\int_s^t A(\tau)d\tau\right)^{k-1}}{(k-1)!}A(s)X_0 ds$$

for all $t\in[0,T]$.

Remark. If the matrix $A(t)\equiv A$, then

$$X(t)=X_0+\sum_{k=1}^{\infty}\left[\frac{A^k t^k}{k!}X_0\right]$$

for all $t\in[0,T]$.

Remark. If the singular values of the matrix A are such that $\sigma_1=\ldots=\sigma_n=\sigma$ and $AX_0=\sigma X_0$, then

$$X(t)=e^{\sigma t}X_0$$

for all $t\in[0,T]$.

References

[1] V. Lakshmikantham, T. Granna Bhaskar, and J. Vasundhara Devi, "Theory of set differential equations in metric spaces," Cambridge Scientific Publishers, Cambridge, 2006.

[2] N.A. Perestyuk, V.A. Plotnikov, A.M. Samoilenko, and N.V. Skripnik, "Differential equations with impulse effects: multivalued right-hand sides with discontinuities," de Gruyter Stud. Math., vol. 40, Berlin/Boston, Walter De Gruyter GmbH& Co, 2011.

[3] V.A. Plotnikov, A.V. Plotnikov, and A.N. Vityuk, "Differential equations with a multivalued right-hand side. Asymptotic methods," AstroPrint, Odessa, 1999.

[4] A.V. Plotnikov and N.V. Skripnik, "Differential equations with "clear" and fuzzy multivalued right-hand side. Asymptotics methods," AstroPrint, Odessa, 2009.

[5] A.A. Martynyuk, "Qualitative analysis of set-valued differential equations." Springer Nature, 2019. https://doi.org/10.1007/978-3-030-07644-3

[6] A. Tolslonogov, "Differential inclusions in a Banach space," Kluwer Publishers, Dordrecht, 2000.

[7] M. Hukuhara, "Integration des applications mesurables dont la valeur est un compact convexe, " Funkc. Ekvacioj, Ser. Int., no. 10, 1967, pp. 205–223